MAA
AMC AMERICAN MATHEMATICS COMPETITIONS

MAA American Mathematics Competitions

43rd Annual

AIME II

American Invitational Mathematics Examination II

Thursday, February 13, 2025

A combination of your AIME score and your AMC 10/12 score is used to determine eligibility for participation in the USA (Junior) Mathematical Olympiad.

INSTRUCTIONS | 说明

1. DO NOT TURN TO THE NEXT PAGE UNTIL YOUR COMPETITION MANAGER TELLS YOU TO BEGIN.
   在未收到监考老师开考指示前，请不要翻开此封面。
2. This is a 15-question competition. All answers are integers ranging from 000 to 999, inclusive.
   这是一套包括15道题目的竞赛，每道题目的答案都是从000到999的整数（包括000和999）。
3. Mark your answer to each problem on the answer sheet with a #2 pencil. Check blackened answers for accuracy and erase errors completely. Only answers that are properly marked on the answer sheet will be scored.
   请将每道题目的答案用＃2铅笔标注在答题卡上。请注意检查涂写的黑色圆圈的准确性，用橡皮完全擦掉错误的答案。只有恰当标注在答题卡上的答案才会被评分。
4. SCORING: You will receive 1 point for each correct answer, 0 points for each problem left unanswered, and 0 points for each incorrect answer.
   评分：每道题目答对得1分， 不答得0分，答错得0分。
5. Only blank scratch paper, rulers, compasses, and erasers are allowed as aids. Prohibited materials include calculators, smartwatches, phones, computing devices, protractors, and graph paper.
   只能使用空白的草稿纸、尺子、圆规和橡皮作为辅助工具。不允许的器具包括计算器、智能手表、手机、计算设备、量角器、坐标纸。
6. Figures are not necessarily drawn to scale.
   图形不一定按比例绘制。
7. Please bubble in carefully the mobile phone number used for registration. This mobile phone number will be used as your unique identification number. The wrongly filled mobile phone number could lead to invalid test score, and such consequence has to be borne by the participant.
   请在答题卡上填涂报名所用手机号码，该手机号码将作为考生唯一识别标志，请务必认真填写，由于号码错误导致成绩无效，后果自负。
8. You will have 3 hours to complete the competition once your competition manager tells you to begin.
   监考老师宣布开始后，你将有3小时的时间完成竞赛。
9. All the problems are written in both English and Chinese. In case of any discrepancy between the two versions, the English version shall prevail.
   所有的问题同时以英文和中文表述。如果两种版本之间存在差异，以英文版本为准。

1. Find the sum of all positive integers n such that $n+3$ divides the product $4(n+4)(n^2+20)$.

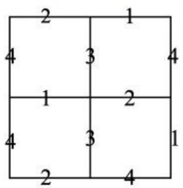
   求所有这样正整数 n 的和，使得 $n+3$ 能整除乘积 $4(n+4)(n^2+20)$.

2. Six points $A, B, C, D, E,$ and F lie in a straight line in that order. Suppose that G is a point not on the line and that $AC=26, BD=22, CE=31, DF=33, AF=73, CG=40,$ and $DG=30$. Find the area of $\triangle AGE$.

   A，B，C，D，E，F 是按此顺序排列在一条直线上的六个点．假设 G 是不在这条直线上的点，并且 $AC=26$，$BD=22$，$CE=31$，$DF=33$，$AF=73$，$CG=40$，$DG=30$．求三角形 $\triangle AGE$ 的面积．

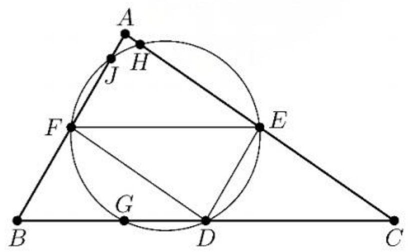
3. Four unit squares form a 2×2 grid. Each of the 12 unit line segments forming the sides of the squares is labeled with one of the four numbers 1, 2, 3, or 4, such that the boundary of each unit square contains exactly one occurrence of each number 1, 2, 3, and 4. One example is shown below. Let N be the total number of such labelings. Find the remainder when N is divided by 1000.

   四个单位正方形构成了 2×2 的网格．构成这些正方形边的 12 条单位线段的每一条都写上 1，2，3，4 这四个数中的一个，使得每个单位正方形的边界上数 1，2，3，4 恰好各出现一次．下图给出了一种符合要求的标数方案的示例．设 N 是这样的标数方案的总数．求 N 除以 1000 的余数．



4. Suppose $\triangle ABC$ has angles $\angle BAC=86^\circ, \angle ABC=60^\circ,$ and $\angle ACB=34^\circ$. Let $D, E,$ and F be the midpoints of sides $\overline{BC}, \overline{AC},$ and $\overline{AB}$, respectively. The circumcircle of $\triangle DEF$ intersects $\overline{BD}, \overline{AE},$ and $\overline{AF}$ at points $G, H,$ and J, respectively. The points $G, D, E, H, J,$ and F divide the circumcircle of $\triangle DEF$ into six minor arcs, as shown. Find $\overarc{DE}+2\cdot\overarc{HJ}+3\cdot\overarc{FG}$, where the arcs are measured in degrees.

   在 $\triangle ABC$ 中，$\angle BAC=86^\circ$，$\angle ABC=60^\circ$，$\angle ACB=34^\circ$．D，E，F 分别是边 $\overline{BC}$，$\overline{AC}$，$\overline{AB}$ 的中点．三角形 DEF 的外接圆与 $\overline{BD}$，$\overline{AE}$，$\overline{AF}$ 在点 G，H，J 处相交．如图所示，点 G，D，E，H，J，F 将三角形 DEF 的外接圆分成六段小圆弧．求 $\overarc{DE}+2\cdot\overarc{HJ}+3\cdot\overarc{FG}$ 的度数．



5. The product is equal to $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

   下面的乘积等于 $\frac{m}{n}$，其中 m 和 n 是互质的正整数．求 $m+n$ 的值．

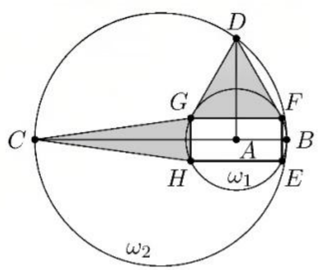
$$\prod_{k=3}^{80}\frac{\log_k\left(5^{k^2-1}\right)}{\log_{k+1}\left(5^{k^2-4}\right)}=\frac{\log_3\left(5^8\right)}{\log_4\left(5^5\right)}\cdot\frac{\log_4\left(5^{15}\right)}{\log_5\left(5^{12}\right)}\cdot\frac{\log_5\left(5^{24}\right)}{\log_6\left(5^{21}\right)}\cdots\frac{\log_{80}\left(5^{6399}\right)}{\log_{81}\left(5^{6396}\right)}$$

6. Let A be the set of positive integer divisors of 2025. Let B be a randomly selected subset of A. The probability that B is a nonempty set with the property that the least common multiple of its elements is 675 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

设 A 为 2025 的正整数约数构成的集合．设 B 为随机选择的 A 的子集．B 是一个非空集合且其元素的最小公倍数是 675 的概率为 $\frac{m}{n}$，其中 m 和 n 是互质的正整数．求 $m+n$ 的值．

7. Circle ω_1 with radius 6 centered at point A is internally tangent at point B to circle ω_2 with radius 15 . Points C and D lie on ω_2 such that $\overline{BC}$ is a diameter of ω_2 and $\overline{BC} \perp \overline{AD}$. The rectangle $EFGH$ is inscribed in ω_1 such that $\overline{EF} \perp \overline{BC}$, C is closer to $\overline{GH}$ than to $\overline{EF}$, and D is closer to $\overline{FG}$ than to $\overline{EH}$, as shown. Triangles $\triangle DGF$ and $\triangle CHG$ have equal areas. The area of rectangle $EFGH$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m-n$.

圆 ω_1 的半径为 6，圆心为 A，与半径为 15 的圆 ω_2 在点 B 处内切．点 C 和 D 位于圆 ω_2 上，使得 $\overline{BC}$ 是圆 ω_2 的直径，并且 $\overline{BC} \perp \overline{AD}$．如图所示，矩形 $EFGH$ 内接于圆 ω_1，使得 $\overline{EF} \perp \overline{BC}$，$C$ 更靠近 $\overline{GH}$，而不是 $\overline{EF}$，并且 D 更靠近 $\overline{FG}$，而不是 $\overline{EH}$．三角形 $\triangle DGF$ 和三角形 $\triangle CHG$ 的面积相等．矩形 $EFGH$ 的面积为 $\frac{m}{n}$，其中 m 和 n 是互质的正整数．求 $m-n$ 的值．



8. From an unlimited supply of 1-cent coins, 10-cent coins, and 25-cent coins, Silas wants to find a collection of coins that has a total value of N cents, where N is a positive integer. He uses the so-called greedy algorithm, successively choosing the coin of greatest value that does not cause the value of his collection to exceed N. For example, to get 42 cents, Silas will choose a 25-cent coin, then a 10-cent coin, then 7 1-cent coins. However, this collection of 9 coins uses more coins than necessary to get a total of 42 cents; indeed, choosing 4 10-cent coins and 2 1-cent coins achieves the same total value with only 6 coins.

In general, the greedy algorithm succeeds for a given N if no other collection of 1-cent, 10-cent, and 25 cent coins gives a total value of N cents using strictly fewer coins than the collection given by the greedy algorithm. Find the number of values of N between 25 and 1000 inclusive for which the greedy algorithm succeeds.

从数量充足的 1 美分硬币、10 美分硬币和 25 美分硬币中，Silas 想找出一组硬币，使其总值为 N 美分，其中 N 是正整数．他使用所谓的贪心算法，在保证硬币总价值不超过 N 美分的前提下，依次选择最大面值的硬币．例如，要得到 42 美分，Silas 会先选择一个 25 美分硬币，然后是一个 10 美分硬币，然后是 7 个 1 美分硬币．然而，形成这样的总价值并不需要有 9 个硬币那么多，如果选择 4 个 10 美分和 2 个 1 美分硬币，仅用 6 个硬币也能达到相同的总价值．

一般来说，对于给定的 N，所谓贪心算法是成功的是指：如果没有其他总价值为 N 美分的 1 美分，10 美分，25 美分的硬币组合使用严格少于贪心算法给出的组合的硬币数量．当 N 从 25 到 1000（包括 25 和 1000）取值时，求贪心算法成功的总次数．

9. Let S be the set of vertices of a regular 24-gon $\mathcal{P}$. Find the number of ways to draw 12 diagonals of $\mathcal{P}$ with equal lengths so that each vertex in S is an endpoint of exactly one of the 12 diagonals.

设 S 为正 24 边形 $\mathcal{P}$ 的顶点构成的集合．现在要画出 12 条相等长度的 $\mathcal{P}$ 的对角线，使得 S 中的每个顶点恰好是这 12 条对角线中某一条的一个端点．求满足上述条件的画法的总数．

10. There are n values of x in the interval $0 \leq x < 2\pi$ where $f(x) = \sin(7\pi \cdot \sin(5x)) = 0$. For t of these n values of x, the graph of $y = f(x)$ is tangent to the x-axis. Find $n + t$.

在区间 $0 \leq x < 2\pi$ 中，有 n 个值 x 使得 $f(x) = \sin(7\pi \cdot \sin(5x)) = 0$．在这 n 个值中有 t 个值，使得 $y = f(x)$ 的图像与 x 轴相切．求 $n + t$ 的值．

11. Fourteen chairs are arranged in a row. Seven people each select a chair in which to sit so that no person sits next to two other people. Let N be the number of subsets of the 14 chairs that could be selected. Find the remainder when N is divided by 1000 .

十四把椅子排成一排．七个人各自选择一把椅子坐下，要求没有人同时与两个其他人相邻而坐，有人就坐的椅子构成 14 把椅子的一个子集．设所有这样的子集的数量为 N．求 N 除以 1000 的余数．

12. Let $A_1A_2A_3\ldots A_{11}$ be an 11-sided non-convex simple polygon with the following properties:

- For every integer $2 \leq i \leq 10$, the area of $\triangle A_iA_1A_{i+1}$ is equal to 1 .
- For every integer $2 \leq i \leq 10, \cos(\angle A_iA_1A_{i+1}) = \frac{12}{13}$.
- The perimeter of the 11-gon $A_1A_2A_3\ldots A_{11}$ is equal to 20 .

Then $A_1A_2 + A_1A_{11} = \frac{m\sqrt{n}-p}{52}$, where m, n, and p are positive integers, and n is not divisible by the square of any prime. Find $m + n + p$.

设 $A_1A_2A_3\ldots A_{11}$ 是一个 11 边形，具有以下性质：

- 对于每个整数 $2 \leq i \leq 10$，三角形 $\triangle A_iA_1A_{i+1}$ 的面积等于 1．
- 对于每个整数 $2 \leq i \leq 10$，$\cos(\angle A_iA_1A_{i+1}) = \frac{12}{13}$．
- 11 边形 $A_1A_2A_3\ldots A_{11}$ 的周长等于 20．

计算可得 $A_1A_2 + A_1A_{11} = \frac{m\sqrt{n}-p}{52}$，其中 m，n，p 为正整数，n 不被任何质数的平方整除．求 $m + n + p$ 的值．

13. Let $\triangle ABC$ be a right triangle with $\angle A = 90°$ and $BC = 35$. There exist points K and L inside the triangle such that

$$AK = AL = BK = CL = KL = 11.$$

The area of the quadrilateral $BKLC$ can be expressed as $n\sqrt{3}$ for some positive integer n. Find n.

设 $\triangle ABC$ 是一个直角三角形，$\angle A = 90°$ 且 $BC = 35$．在三角形内部存在点 K 和 L ，使得

$$AK = AL = BK = CL = KL = 11\,.$$

四边形 $BKLC$ 的面积可以表示为 $n\sqrt{3}$，其中 n 为正整数．求 n 的值．

14. There are exactly three positive real numbers k such that the function

$$f(x)=\frac{(x-27)(x-48)(x-108)(x-k)}{x}$$

defined over the positive real numbers achieves its minimum value at exactly two positive real numbers x. Find the sum of these three values of k.

恰好存在三个正实数 k，使得定义在正实数集上的函数

$$f(x)=\frac{(x-27)(x-48)(x-108)(x-k)}{x}$$

的最小值恰好在两个正实数 x 处取到. 求这三个 k 的值之和.

15. Let $x_1, x_2, x_3, \ldots$ be a sequence of rational numbers defined by $x_1=\frac{25}{11}$ and

$$x_{k+1}=\frac{1}{3}\left(x_k+\frac{1}{x_k}-1\right)$$

for all $k \geq 1$. Then x_{2025} can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find the remainder when $m+n$ is divided by 1000 .

设 x_1，x_2，x_3，$\ldots$ 是按如下方式定义的有理数序列：$x_1=\frac{25}{11}$，并且对于所有 $k \geq 1$ 的正整数

$$x_{k+1}=\frac{1}{3}\left(x_k+\frac{1}{x_k}-1\right).$$

设 x_{2025} 可以表示为 $\frac{m}{n}$，其中 m 和 n 是互质的正整数. 求 $m+n$ 除以 1000 的余数.